An expansion for Neutrino Phenomenology

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We develop a formalism for constructing the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix and neutrino masses using an expansion that originates when a sequence of heavy right handed neutrinos are integrated out, assuming a seesaw mechanism for the origin of neutrino masses. The expansion establishes relationships between the structure of the PMNS matrix and the mass differences of neutrinos, and allows symmetry implications for measured deviations from tri-bimaximal form to be studied systematically. Our approach does not depend on choosing the rotation between the weak and mass eigenstates of the charged lepton fields to be diagonal. We comment on using this expansion to examine the symmetry implications of the recent results from the Daya-Bay collaboration reporting the discovery of a non zero value for θ_{13}, indicating a deviation from tri-bimaximal form, with a significance of $5.2\,\sigma$.

I. INTRODUCTION

The standard theory of neutrino oscillations, where three mass eigenstate neutrinos differ from their interaction eigenstates leading to the observed neutrino oscillations, is consistent with current experimental data. The amplitude of the oscillations among various neutrino species is related to the misalignment of the interaction and mass eigenstates of the neutrinos, characterized by the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) matrix [1, 2]. Taking the relation between the interaction (primed) and mass (unprimed) eigenstates to be given by $f'_{L/R} = \mathcal{U}(f, L/R)\, f_{L/R}$, where $f = \{\nu, e, \mu, \cdots\}$, the PMNS matrix is given by

$$\mathcal{U}_{PMNS} = \mathcal{U}^\dagger(e, L)\,\mathcal{U}(\nu, L). \tag{1}$$

The $\mathcal{U}_{PMNS}$ matrix can be parameterized in terms of three angles $\theta_{12}, \theta_{13}, \theta_{23}$ and three CP violating phases $\delta, \alpha_{1,2}$. Defining $s_{ij} = \sin\theta_{ij}$ and $c_{ij} = \cos\theta_{ij}$ with the conventions $0 \leq \theta_{ij} \leq \pi/2$, and $0 \leq \delta, \alpha_{1,2} \leq 2\pi$

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a general parameterization of this matrix is given by

$$\mathcal{U}_{PMNS} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \times \begin{pmatrix} c_{13} & 0 & s_{13}e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{-i\delta} & 0 & c_{13} \end{pmatrix} \times \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} P, \tag{2}$$

where $P = \text{diag}(e^{i\alpha_1/2}, e^{i\alpha_2/2}, 1)$ is a function of the Majorana phases, present if the right handed neutrinos are Majorana, while δ is a Dirac phase. This latter phase can contribute in principle to neutrino oscillation measurements, while the Majorana phases cannot. Recent global fit results on neutrino mass differences and measured mixing angles using old/new reactor fluxes are given in Ref. [3], (with new reactor flux results in brackets):

$$\Delta m_{21}^2 = (7.58^{+0.22}_{-0.26}) \times 10^{-5} \text{eV}^2,$$

$$|\Delta \bar{m}_{32}^2| = (2.35^{+0.12}_{-0.09}) \times 10^{-3} \text{eV}^2,$$

$$\sin^2(\theta_{12}) = 0.306^{+0.18}_{-0.15} \quad (0.312^{+0.17}_{-0.16}),$$

$$\sin^2(\theta_{13}) = 0.021^{+0.007}_{-0.008} \quad (0.025^{+0.007}_{-0.007}),$$

$$\sin^2(\theta_{23}) = 0.42^{+0.08}_{-0.03}. \tag{3}$$

The error is the reported 1σ error. Note that $\Delta \bar{m}_{32}^2 \equiv m_3^2 - (m_1^2 + m_2^2)/2$ and $\Delta \bar{m}_{32}^2 > 0, (< 0)$ corresponds to a normal (inverted) mass spectrum. This pattern of experimental data is perhaps suggestive of a PMNS matrix that has at least an approximate "tri-bimaximal" ($\mathcal{T}B$) form [4]. Fixing $\sin^2(\theta_{12}) = 1/3$ and $\theta_{13} = 0$ the $\mathcal{T}B$ form is

$$\mathcal{U}_{PMNS} \approx \mathcal{U}_{TB} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \tag{4}$$

for a particular phase convention.

The structure of this matrix could be fixed by underlying symmetries. In attempting to determine such an origin of this matrix, the "flavour" basis where one assumes $\mathcal{U}(e, L) = \text{diag}(1, 1, 1)$ is frequently used. When this assumption is employed the relationship between the weak and mass neutrino eigenstates is identified with the $\mathcal{U}_{PMNS}$, i.e. $\nu_i' = (\mathcal{U}_{PMNS})_{ij} \nu_j$ for $i, j = 1, 2, 3$. There have been many attempts to link the approximate $\mathcal{T}B$ form of the neutrino mass matrix to symmetries of the right handed neutrino interactions in this basis, see Ref. [5, 6] for a review. Recent experimental results provide evidence for deviations from this $\mathcal{T}B$ form. Evidence for $\sin^2(\theta_{13}) \neq 0$ in global fits is reported to be $> 3\sigma$ in Ref. [3] at this time. As this paper was approaching completion, the discovery of non-vanishing θ_{13} was announced

by the Daya Bay Collaboration [7] with a reported value of

$$\sin^2(2\,\theta_{13}) = 0.092 \pm 0.016(\text{stat}) \pm 0.005(\text{syst}), \tag{5}$$

corresponding to 5.2σ evidence for non zero θ_{13}. It is reasonable to expect further speculation about the origin of the deviation from $\mathcal{T}B$ form in light of this result, where again the flavour basis will be frequently assumed.

There is no clear experimental support for assuming that $\mathcal{U}(e, L) = \text{diag}(1, 1, 1)$. This choice can be motivated by an ansatz related to the origin of the approximately diagonal structure of the Cabibbo-Kobayashi-Maskawa (CKM) matrix, and a further ansatz on the relation between $\mathcal{U}^\dagger(d, L)$ and $\mathcal{U}^\dagger(e, L)$, see Ref. [5] for a coherent discussion on this approach. This choice can also be justified with model building in principle, see Ref. [8] for an example. Conversely, in grand unified models frequently associated with the high scale involved in the seesaw mechanism, the quark and lepton mass matricies can be related in such a manner that the flavour basis cannot be chosen, or at least, the choice of the flavour basis can be highly artificial.

It is of interest to have a formalism for neutrino phenomenology that is as robust and basis independent as possible. Clearly linking symmetries to the form of the PMNS matrix requires that the physical conse-quences of a symmetry are not dependent on a basis that can be arbitrarily choosen for $\mathcal{U}(e, L)$, such as the flavour basis. In this paper, we develop a perturbative approach to the structure of the PMNS matrix as an alternative to symmetry studies that are basis dependent.[1] Using this approach we show a basis independent relationship between the eigenvectors that give $\mathcal{U}(\nu, L)$ and $\mathcal{U}(e, L)$ corresponding to $\mathcal{T}B$ form at leading order in our expansion. We also examine the prospects of relating patterns in the data in low scale measure-ments of neutrino properties with high scale flavour symmetry breaking, illustrating how our approach can be used to study the recent reported discovery of non zero θ_{13} reported in Ref. [7].

II. CONSTRUCTING A FLAVOUR SPACE EXPANSION FOR NEUTRINO PHENOMENOLOGY

In this section we develop a formalism linking the measured differences in the neutrino mass eigenstates with the structure of the PMNS matrix, which we will refer to as a flavour space expansion (FSE). There is an experimental ambiguity in the measured mass hierarchies at this time. The neutrino mass spectrum can be a normal hierarchy ($m_3 > m_2 \gtrsim m_1$) or an inverted hierarchy ($m_3 < m_1 \lesssim m_2$). In our initial discussion we will assume a normal hierarchy. The formalism can be reinterpreted for an inverted hierarchy.

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[1] This is a modern implementation in the neutrino sector of an old idea of relating the mixing matricies of the standard model (SM) to the measured quark or neutrino masses, for pioneering studies with this aim see Ref. [9, 14–16].

A. Review of Seesaw the Mechanism

Recall the standard seesaw scenario [17], with three right handed neutrinos N_{Ri}.[2] We use the notation $\ell = (\nu_L, e_L)$ for the left handed SU(2) doublet field and e_R for the SU(2) singlet lepton field carrying hypercharge. These fields carry flavour indicies i, j.[3] We also define $\tilde{H} = i\tau_2 H^\star$, where H is the Higgs doublet of the SM, with $\langle H^T \rangle = (0, v/\sqrt{2})$. Then the lepton sector of the Lagrangian in the seesaw scenario is the following

$$\mathcal{L} = i\,\bar{N}'_{Ri}\,\slashed{\partial}\,N'_{Ri} - \frac{1}{2}\,\overline{N'^c_{Ri}}\,M'_{ij}\,N'_{Rj} - \bar{\ell}'_{Li}\,H\,(y'_E)_{ij}\,e'_{Rj} - \bar{N}'_{Ri}\,(y'_\nu)_{ij}\,\tilde{H}^\dagger\,\ell'_{Lj} + h.c.\ , \tag{6}$$

here we have defined $N^c_{Ri} = C\bar{N}'^T_{Ri}$ with C the charge conjugation matrix. There is freedom to rotate to the mass basis by introducing the unitary rotation matricies $\mathcal{U}(f, L/R)$, so that

$$N'_{Ri} = \mathcal{U}(N,R)_{ij}\,N_{Rj}, \qquad \ell'_{Li} = \mathcal{U}(L,L)_{ij}\,\ell_{Lj}, \qquad e'_{Ri} = \mathcal{U}(e,R)_{ij}\,e_{Rj}. \tag{7}$$

The kinetic terms are unchanged by these rotations, and one is free to further rotate to a basis in the flavour space of M, y_E, y_ν defined by the set of all possible M, y_E, y_ν related through

$$\begin{aligned}
M' &\to \mathcal{U}(N,R)^\star\,M\,\mathcal{U}(N,R)^\dagger, \\
y'_E &\to \mathcal{U}(L,L)\,y_E\,\mathcal{U}(e,R)^\dagger, \\
y'_\nu &\to \mathcal{U}(N,R)\,y_\nu\,\mathcal{U}(L,L)^\dagger.
\end{aligned} \tag{8}$$

We choose to make the initial rotation to a basis where M is diagonal real and nonnegative. This fixes $\mathcal{U}(N,R)$. Initially the Majorana mass matrix has three complex eigenvalues $M_{a,b,c} = \eta_{a,b,c}\,|M_{a,b,c}| = \eta_{a,b,c}\,(\mu_{a,b,c})$, here $\eta_{a,b,c}$ are the Majorana phases. Working in this basis [18–20] shifts any Majorana phases into the y_ν matrix and $N_i = \sqrt{\eta_i}\,N_{Ri} + \sqrt{\eta_i^\star}\,N^c_{Ri}$. Integrating out the heavy N_i one obtains the dimension five operator [21]

$$\mathcal{L}_5 = \frac{1}{2}\,(\tilde{H}^\dagger \ell_i)\,C_{ij}\,(\tilde{H}^\dagger \ell_j) + h.c.\ , \tag{9}$$

with the matrix Wilson coefficient $C_{ij} = (y_\nu^T\,\eta M^{-1}\,y_\nu)_{ij}$. The key observation that we use in this work is that when integrating out the N_i in sequence[4] a perturbative expansion for neutrino phenomenology that is

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[2] Our initial discussion will largely follow Ref. [18–20].
[3] These flavour indices will run over a, b, c for the three N'_i. We use this notation to distinguish these flavours from the measured mass differences of the physical eigenstates ($m_{1,2,3}$) and also the index on the Yukawa vectors of each N'_i, which run over $1, 2, 3$.
[4] We distinguish in this paper right handed neutrinos that are in a mass diagonal basis, reserving the notation N_i for such states, compared to N_{Ri} states, which are not mass diagonal in general.

related to a hierarchy in the magnitude of the contributions to C_{ij} can be constructed. Note that this is also the key point in the sequential dominance idea of Refs. [10–13], however, the formalism we will develop is distinct from these past results.

B. Perturbing the Seesaw

1. Model Dependence of Perturbing the Seesaw

The FSE we develop depends on the mass spectrum and the Yukawa couplings of the N_i. Also, in principle, the flavour orientation[5] effects the quality of the FSE. Experimentally, at this time, all of these Lagrangian parameters are individually unknown, thus we will be forced to assume that a perturbative expansion of the form we employ exists. Our approach is to view the seesaw Lagrangian given in Eq. (6) as an effective theory, and we do not attempt to uniquely identify and restrict ourselves to a particular UV physics that dictates the low energy parameters in this effective theory in this paper. However, the formalism we develop allows perturbative investigations of the neutrino mass spectrum and mixing angles in many scenarios and is in fact quite general. It would be surprising if all of the unknown parameters conspired to forbid an expansion of this form from being present. We seek to illustrate this point in this section, by demonstrating a simple scenario where the FSE would be of some utility.

There are of course cases when the formalism we outline cannot be used. When the $\mu_{a,b,c}$ and Yukawa couplings of the N_i are both highly degenerate a FSE cannot be used.[6] This is also the case if a non-degenerate pattern of the $\mu_{a,b,c}$ and the Yukawa couplings are such that the perturbations to the neutrino masses are similar in magnitude. However, as the $\mu_{a,b,c}$ depend on the matching of the effective theory to UV physics, while the Yukawa couplings are dimension four and are not UV sensitive, this would require tuning the physics of different energy scales. Unless model building is used to relate the masses and Yukawa couplings of the N_i so that the contributions to the ν_L masses are the same, a FSE is expected quite generally, and can be related to neutrino mass differences, as we will show.

One can study the expected spectrum of left handed masses due to Eqn. (6) in generic scenarios. Without assuming other interactions beyond the SM, the N'_{Ri} are not distinguished by any quantum number. As a result, the non-diagonal mass matrix that is the coefficient of the operator matrix $\mathcal{O}_{ij} = N'_{Ri} N'_{Rj}$ given by M'_{ij} is naively expected to have entries that are all similar in magnitude. As the operator is dimension

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[5] By the flavour orientation of the Yukawa couplings, we mean the orientation of the Yukawa coupling vectors $\vec{x}, \vec{y}, \vec{z}$ with respect to the leading order eigenvectors $\vec{\rho}_{a,b,c}$; see the next section for definitions and further details.

[6] An exception to this statement is when the flavour orientation is such that an expansion is present due to the geometry in flavour space, we discuss an example of this form in Section II D.

three, the mass matrix is expected to be proportional to the highest scale UV physics (violating lepton number) that was integrated out leading to this effective theory. We denote this scale by M_0 and the naive non-diagonalized mass matrix in this case as

$$M'_{ij} \simeq M_0 \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \tag{10}$$

with eigenvalues $\mu_i^0 = \{3\,M_0, 0, 0\}$. The vanishing of the two eigenvalues of M'_{ij} can be lifted by interactions of the N'_{Ri}. These interactions can be dictated by beyond the SM quantum numbers assigned in UV model building. If the mass matrix is still approximately degenerate, as in Eq.(10), then a hierarchy of the N_i masses is still expected. An example of a small breaking where other interactions do not need to be assumed is given by the orientations of the N'_{Ri} in flavour space. Rotating to the lepton diagonal mass basis, these interactions give loop corrections

$$\epsilon_{ij} = \delta M'_{ij}/M'_{ij} \simeq \frac{(y'_\nu)^*_{ik}(y'_\nu)_{jk}}{16\,\pi^2} \log\left(\frac{\mu^2}{M_0^2}\right) \qquad \text{no sum on } i, j. \tag{11}$$

Logarithmic corrections of this form are required to cancel the renormalization scale dependence of the pole masses of the N'_{Ri}. Including such corrections leads to $M'_{ij} \simeq M_0\,(1 + \epsilon_{ij})$. These corrections split the mass spectrum; diagonalizing one finds

$$\mu_1 = M_0\left(3 + \frac{\sum_{ij}\epsilon_{ij}}{3}\right), \quad \mu_2 = \frac{2}{3}\,M_0\left[\sum_i \epsilon_{ii} - \sum_{i<j}\epsilon_{ij}\right], \quad \mu_3 = M_0\,\mathcal{O}(\epsilon^2). \tag{12}$$

A hierarchical spectrum of μ_i is expected with the pattern $(\mu_1, \mu_2, \mu_3) \simeq M_0(1, \epsilon, \epsilon^2)$.

In the case of degenerate μ_i due to the M'_{ij} mass matrix not conforming to these generic expectations, the expansion can follow from a hierarchy in the Yukawa couplings of the N_i – such as the hierarchical pattern of Yukawa couplings in the SM. Lastly, the suitability of the expansion can follow only from the orientation in flavour space of the Yukawa coupling vectors of the N_i. In summary, the appropriateness of the FSE is clearly model dependent, but we expect it to be broadly applicable in realistic models.

C. Developing the Seesaw Perturbations

Consider supplementing the SM field content by a single right handed neutrino[7] N_a and an interaction term that couples it into a linear combination of ℓ_L^β. The coupling is fixed by the complex Yukawa vector

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[7] The subscript in N_i will run over labels a, b, c; for now we are just considering the single spinor field N_a.

$\vec{x}^T = \{x_1, x_2, x_3\}$ in flavour space whose form can be constrained by a flavour symmetry, but is here left arbitrary. We absorb the overall Majorana phase into this vector. The relevant Lagrangian terms[8] are given by

$$\mathcal{L}_a = -\frac{\mu_a}{2} N_a N_a - x_\beta \bar{N}_a \tilde{H}^\dagger \ell_L^\beta + h.c. \tag{13}$$

The right handed neutrino can be integrated out, giving for the left handed neutrino mass matrix a nonzero eigenvalue

$$\mathcal{M} \equiv \frac{v^2}{2\,\mu_a} \vec{x}\,\vec{x}^T = \mathcal{U}(\nu, L)^\star \operatorname{diag}(0, 0, m_a) \mathcal{U}(\nu, L)^\dagger. \tag{14}$$

The matrix $\mathcal{U}(\nu, L)$ is the matrix $(\vec{\rho}_c^{\,\star}, \vec{\rho}_b^{\,\star}, \vec{\rho}_a^{\,\star})$ of normalized (column) vectors $\vec{\rho}_{a,b,c}^{\,\star}$ that solve $\mathcal{M}\vec{\rho}^{\,\star} = m\vec{\rho}$, with m real and non-negative.[9] These vectors are also eigenvectors of $\mathcal{M}^\dagger \mathcal{M} = (\vec{x}^\star \vec{x}^T)(\vec{x}\,\vec{x}^T) = (\vec{x}^\dagger \vec{x})\vec{x}^\star \vec{x}^T$ with eigenvalues m^2; their complex conjugates, $\vec{\rho}$ are eigenvectors of $\mathcal{M}\,\mathcal{M}^\dagger = (\vec{x}\vec{x}^T)(\vec{x}^\star \vec{x}^\dagger) = (\vec{x}^\dagger \vec{x})\vec{x}\vec{x}^\dagger$, also with eigenvalues m^2. One finds the leading eigenvector and eigenvalue

$$\vec{\rho}_a = \vec{x}/|\vec{x}|, \qquad m_a = v^2(\vec{x}^\dagger \vec{x})/2\mu_a. \tag{15}$$

Here μ_a is real and non-negative due to the initial flavour basis choice that fixed $\mathcal{U}(N, R)$, while $\mathcal{U}(\nu, L)$ and $\vec{x}$ are in general complex.[10] The remaining two eigenvectors $\vec{\rho}_{b,c}$ are such that $\langle \vec{\rho}_a | \vec{\rho}_b \rangle \equiv \vec{\rho}_a^{\,\dagger} \vec{\rho}_b = 0$, $\langle \vec{\rho}_a | \vec{\rho}_c \rangle = 0$. These eigenvectors will lead to the remaining (smaller) mass eigenvalues, and will perturb the leading eigenvalue and eigenvector and thus the $\mathcal{U}(\nu, L)$ matrix. This is the expansion we seek to exploit. Consider the perturbation that generates the second eigenvalue of the neutrino mass matrix due to a second right handed neutrino N_b, which we define to couple into a linear combination of the ℓ_L given by $\vec{y}^T = \{y_1, y_2, y_3\}$. One obtains a second eigenvalue in the left handed neutrino mass matrix so long as $\vec{y} \not\parallel \vec{x}$. The perturbation of $\mathcal{M}\,\mathcal{M}^\dagger$ is given by

$$
\begin{aligned}
(\mathcal{M} + \delta\mathcal{M})(\mathcal{M} + \delta\mathcal{M})^\dagger &= \frac{v^4}{4\mu_a^2}\, \vec{x}\,\vec{x}^T\,(\vec{x}^\star \vec{x}^\dagger) + \frac{v^4}{4\mu_a\,\mu_b}(\vec{x}\,\vec{x}^T)(\vec{y}^\star \vec{y}^\dagger) + \frac{v^4}{4\mu_a\,\mu_b}(\vec{y}\,\vec{y}^T)(\vec{x}^\star \vec{x}^\dagger), \\
&+ \frac{v^4}{4\mu_b^2}\, \vec{y}\,\vec{y}^T\,(\vec{y}^\star \vec{y}^\dagger).
\end{aligned}
\tag{16}
$$

At leading order the $\vec{\rho}_{b,c}$ have degenerate (vanishing) eigenvalues. One is free to rotate to a chosen basis in these vectors. We rotate to a basis in these vectors such that the following perturbation vanishes for $\vec{\rho}_c$

$$\langle \vec{\rho}_c | (yy^\dagger)(y^T y^\star) | \vec{\rho}_c \rangle = 0. \tag{17}$$

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[8] We have chosen $\mathcal{U}(N, R)$ as discussed in Section 6 to rotate to a mass diagonal basis for N_i and absorbed this rotation into a redefinition of the Yukawa matrices y_ν. The Yukawa vectors $\vec{x}$, $\vec{y}$ and $\vec{z}$ introduced below are given in this basis.

[9] Here we use the notation that the eigenvectors run over the flavour indicies a, b, c to denote that they are associated with integrating out each of the $N_{a,b,c}$ at leading order in the corresponding mass eigenvalues.

[10] This choice is allowed by the unitary flavour transformations that are symmetries of the kinetic terms, Eqs. (6–8).

With this choice $\vec{\rho}_c$ retains a vanishing eigenvalue when N_b is integrated out. The eigenvector $\vec{\rho}_c$ should be orthogonal to $\vec{\rho}_a \propto \vec{x}$. A normalized eigenvector basis at leading order is then

$$\vec{\rho}_b = \frac{\vec{x}^\star \times (\vec{y} \times \vec{x})}{|\vec{x}|\,|\vec{x} \times \vec{y}|}, \qquad \vec{\rho}_c = \frac{\vec{y}^\star \times \vec{x}^\star}{|\vec{x} \times \vec{y}|}. \tag{18}$$

Now we can determine the perturbation on the leading order eigenvectors and eigenvalues when N_b is integrated out. The corrections to the eigenvalues and eigenvectors using perturbation theory are given by

$$
\begin{aligned}
\delta\vec{\rho}_j &= -\sum_{i \neq j} \frac{\langle \vec{\rho}_i | \mathcal{M}\delta\mathcal{M}^\dagger + \delta\mathcal{M}\,\mathcal{M}^\dagger | \vec{\rho}_j \rangle}{m_j^2 - m_i^2}\, \vec{\rho}_i, \\
\delta m_i^2 &= \langle \vec{\rho}_i | \mathcal{M}\delta\mathcal{M}^\dagger + \delta\mathcal{M}\,\mathcal{M}^\dagger + \delta\mathcal{M}\,\delta\mathcal{M}^\dagger | \vec{\rho}_i \rangle.
\end{aligned} \tag{19}
$$

Nonzero eigenvalues are obtained for $m_{b,c}$ at second order in the expansion due to the orthogonal basis vectors causing the leading perturbations to each of these masses to vanish. The leading perturbation to the eigenvectors and m_a is first order in the expansion. We retain the leading perturbation on the eigenvectors and the leading and subleading effects on the masses to obtain nonzero eigenvalues. We find the following for the perturbations

$$
\begin{aligned}
\delta\vec{\rho}_a &= \mu_{ab}^2 \frac{\langle \vec{\rho}_b | \vec{y} \rangle (\vec{y} \cdot \vec{x}^\star)|\vec{x}|}{\delta m_{ab}^2}\, \vec{\rho}_b, & \delta m_a^2 &= 2\,\mu_{ab}^2\, \mathrm{Re}\left[\langle \vec{y}|\vec{x}\rangle\,(\vec{x}\cdot\vec{y}^\star)\right] + \mu_{bb}^2\,|\vec{y}|^4\,\cos^2\theta_{xy}, \\
\delta\vec{\rho}_b &= \mu_{ab}^2 \frac{\langle \vec{y}|\vec{\rho}_b\rangle (\vec{x}\cdot\vec{y}^\star)|\vec{x}|}{\delta m_{ba}^2}\, \vec{\rho}_a, & \delta m_b^2 &= \mu_{bb}^2 |\langle \vec{\rho}_b|\vec{y}\rangle|^2\,|\vec{y}|^2, \\
\delta\vec{\rho}_c &= 0, & \delta m_c^2 &= 0.
\end{aligned} \tag{20}
$$

Here $\delta m_{ij}^2 = m_i^2 - m_j^2$, $\mu_{ij}^2 = v^4/(4\,\mu_i\,\mu_j)$ and $\cos\theta_{xy} = |\vec{x}^* \cdot \vec{y}|/|\vec{x}||\vec{y}|$ is a measure of (the cosine of) the angle between the $\vec{x}, \vec{y}$ Yukawa vectors. The masses are evaluated to the appropriate order in the perturbative expansion and the eigenvector perturbations are in general complex. Note that for the phenomenology of the $\mathcal{U}_{PMNS}$ matrix that we will study it will be sufficient to only retain the leading perturbation, while when studying the mass spectrum the leading and sub-leading perturbations should be retained.

Finally integrate out the third right handed neutrino N_c with Majorana mass μ_c, which couples into a linear combination of the ℓ_L^β dictated by $z_\beta^T = \{z_1, z_2, z_3\}$. The resulting eigenvector perturbations are

$$
\begin{aligned}
\delta_2\vec{\rho}_a &= \mu_{ac}^2 \frac{\langle \vec{\rho}_b | \vec{z} \rangle (z \cdot x^\star)|\vec{x}|}{\delta m_{ab}^2}\, \vec{\rho}_b + \mu_{ac}^2 \frac{\langle \vec{\rho}_c | \vec{z} \rangle (z \cdot x^\star)|\vec{x}|}{\delta m_{ac}^2}\, \vec{\rho}_c, \\
\delta_2\vec{\rho}_b &= \mu_{ac}^2 \frac{\langle \vec{z}|\vec{\rho}_b\rangle (x \cdot z^\star)|\vec{x}|}{\delta m_{ba}^2}\, \vec{\rho}_a, \\
\delta_2\vec{\rho}_c &= \mu_{ac}^2 \frac{\langle \vec{z}|\vec{\rho}_c\rangle (x \cdot z^\star)|\vec{x}|}{\delta m_{ca}^2}\, \vec{\rho}_a.
\end{aligned} \tag{21}
$$

The mass perturbations are

$$\delta_2 m_a^2 = 2\,\mu_{ac}^2\,\mathrm{Re}\left[\langle\vec{z}|\vec{x}\rangle\,(\vec{x}\cdot\vec{z}^\star)\right] + \mu_{cc}^2\,|\vec{z}|^4\,\cos^2\theta_{xz},$$
$$\delta_2 m_b^2 = \mu_{cc}^2\,|\langle\vec{\rho_b}|\vec{z}\rangle|^2\,|\vec{z}|^2,$$
$$\delta_2 m_c^2 = \mu_{cc}^2\,|\langle\vec{\rho_c}|\vec{z}\rangle|^2\,|\vec{z}|^2. \tag{22}$$

The measured masses of the neutrino's are related to these perturbations as

$$m_A^2 = m_a^2 + \delta m_a^2 + \delta_2 m_a^2,$$
$$m_B^2 = \delta m_b^2 + \delta_2 m_b^2,$$
$$m_C^2 = \delta_2 m_c^2. \tag{23}$$

It is interesting to note that a normal hierarchy emerges quite naturally from the FSE as the leading neutrino mass m_a receives corrections at linear order to its mass, while the remaining masses only receive corrections at second order in the perturbations.

It is also important to note that this formalism does not require a hierarchy of the form $\delta_2 m_i^2 \ll \delta m_i^2$, only $\delta_2 m_i^2, \delta m_i^2 \ll m_a^2$ is required. Expanding on this important point in more detail, it is not required that the perturbation due to integrating out N_b is larger than the perturbation due to integrating out N_c. Only that the effect of integrating out each of these right handed neutrinos perturbs the initial mass matrix —which is dominated by integrating out the initial right handed neutrino N_a. The existence of these perturbations are not necessarily direct statements on the relative size of the μ_i as we discuss in more detail in the next section.

D. Inverted and Normal Hierarchy and Flavour Space

For a normal hierarchy (with notation $m_3 > m_2 \gtrsim m_1$) we identify $(1,2,3) = (C,B,A)$ in the equations above. The difference between the normal and inverted ($m_3 < m_1 \lesssim m_2$) case appears in the relative size of the δm_i and $\delta_2 m_i$ and the identification $(C,B,A) = (3,1,2)$ for an inverted hierarchy. An inverted hierarchy requires non generic perturbations in the FSE, or one can trivially modify the expansion to only perturb when N_c is integrated out. The size of δm_i and $\delta_2 m_i$ depends on the hierarchy in $\mu_{a,b,c}$, the magnitude of the Yukawa vectors, and also the orientation in flavour space of the vectors $\vec{x}$, $\vec{y}$, $\vec{z}$. In this section, we will discuss the size of the perturbations of the FSE in light of neutrino mass data in a scenario where the perturbations are dictated primarily by a hierarchy in $\mu_{a,b,c}$. We will then discuss the case where the expansion arises primarily due to the orientation of the Yukawa coupling vectors in flavour space.

1. The expansion without flavour alignment

We can examine the quality of the expansion by comparing to the measured mass differences in neutrinos. Consider the case that the size of the perturbations is generic in the sense that the Yukawa coupling vectors are taken to be $\mathcal{O}(1)$ with the orientation in flavour space not significantly effecting the quality of the expansion. In this case, the expansion follows from the relative size of the μ_i, i.e. $\mu_a < \mu_b < \mu_c$. Consider the generic case in a normal hierarchy. Using the FSE and retaining the dominant term

$$\Delta m_{32}^2 \;=\; m_a^2 + \delta m_a^2 + \delta_2 m_a^2 - (\delta m_b^2 + \delta_2 m_b^2) \approx m_a^2, \tag{24}$$

while in the same manner $\delta m_b^2 \sim \Delta m_{21}^2$ and the expansion requires $\delta_2 m_c^2 < \Delta m_{32}^2, \Delta m_{21}^2$. Due to this, the expansion requires $v^2 |\vec{z}|^2/(2\,\mu_c) < \sqrt{\Delta m_{32}^2} \sim 0.05\,\mathrm{eV}$. This condition is consistent with current bounds on the absolute neutrino mass scale, with a 95% C.L. bound of $\sum m_\nu = 0.28\,\mathrm{eV}$ quoted in Ref. [22], assuming ΛCDM cosmology. It is also consistent with current bounds from Tritium β decay experiments [23] which quote $m(\nu_e) < 2\,\mathrm{eV}$ at 95% C.L.

Expressing this condition in terms of the high mass scale of the N_c integrated out, $\mu_c/|\vec{z}|^2 \gtrsim 10^{14}\,\mathrm{GeV}$. Generically one expects the mass scale of the right handed neutrino operator to be the largest scale integrated out that violated L number, and this condition for the lightest neutrino is clearly consistent with naive expectations of $M_0 \sim M_{pl}$.

2. The expansion with flavour alignment

Now consider the case where the perturbative expansion follows from the flavour orientation of the $\vec{x}, \vec{y}, \vec{z}$ vectors primarily. An example where this is the case is when the threshold matching onto the UV physics is such that the Wilson coefficient matrix of $\mathcal{O}_{ij}$ yields a mass matrix with nearly degenerate eigenvalues. This occurs for example when

$$M'_{ij} \simeq M_0 \begin{pmatrix} 1+\epsilon & \epsilon & \epsilon \\ \epsilon & 1+\epsilon & \epsilon \\ \epsilon & \epsilon & 1+\epsilon \end{pmatrix}. \tag{25}$$

In this case, a nearly degenerate mass spectrum of the N_R follows, $\mu_a \simeq \mu_b \simeq \mu_c$. Generating a normal or inverted hierarchy if one also has $|\vec{x}| \sim |\vec{y}| \sim |\vec{z}|$ requires more precise alignments in flavour space and allows a geometric interpretation of the measured neutrino mass spectrum. In the FSE, the tree level masses

of the SM neutrinos are

$$m_A^2 = \mu_{aa}^2 |\vec{x}|^4 + 2\mu_{ab}^2|\vec{x}|^2|\vec{y}|^2 \cos^2\theta_{xy} + 2\mu_{ac}^2|\vec{x}|^2|\vec{z}|^2 \cos^2\theta_{xz} + \mu_{bb}^2|\vec{y}|^4 \cos^2\theta_{xy} + \mu_{cc}^2|\vec{z}|^4 \cos^2\theta_{xz},$$

$$m_B^2 = \mu_{bb}^2|\vec{y}|^4 \cos^2\theta_{\rho_b y} + \mu_{cc}^2|\vec{z}|^4 \cos^2\theta_{\rho_b z},$$

$$m_C^2 = \mu_{cc}^2|\vec{z}|^4 \cos^2\theta_{\rho_c z}. \tag{26}$$

Consider the case that all of the μ_{ij}^2 are similar in magnitude and $|\vec{x}| \gtrsim |\vec{y}| \sim |\vec{z}|$ so that the mass spectrum is primarily dictated by the orientation of the Yukawa vectors in flavour space. An example of an inverted or normal hierarchy is shown in Fig. (1) in this case.
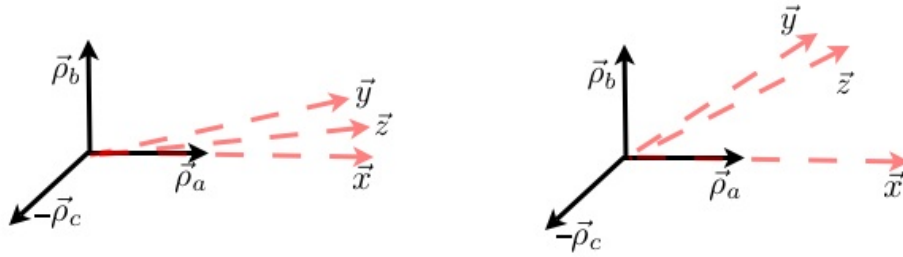

FIG. 1: A geometric interpretation of an inverted or normal hierarchy when $\mu_a \simeq \mu_b \simeq \mu_c$ and $|\vec{x}| \gtrsim |\vec{y}| \sim |\vec{z}|$. Normal hierarchy on the left, inverted on the right.

E. $\mathcal{U}_{PMNS}$ and Flavour Space

Now, assuming that a FSE expansion exists, let us consider its utility in examining the form of $\mathcal{U}_{PMNS}$. The rotation matrix $\mathcal{U}(\nu, L)$ with $\vec{v}_i^\star = (\vec{\rho}_i + \delta\vec{\rho}_i + \delta_2\vec{\rho}_i)^\star$ is given by $\mathcal{U}(\nu, L) = (\vec{v}_3^\star, \vec{v}_2^\star, \vec{v}_1^\star)$. The charged lepton mass matrix after electroweak symmetry breaking is given by $\mathcal{M}_e = v\,y_E/\sqrt{2}$ and diagonalized by[11]

$$\mathcal{U}(e, R)\,\mathcal{M}_e\,\mathcal{U}^\dagger(e, L) = \text{diag}(m_e, m_\mu, m_\tau). \tag{27}$$

---

[11] The hierarchy of the charged lepton masses can be used to organize an expansion of $\mathcal{U}(e, L)$ in the same manner by constructing $\mathcal{M}_e^\dagger \mathcal{M}_e$ in principle. Conversely, in principle, one could employ an ansatz that the hierarchy in these mass eigenvalues could be related to an expansion of $\mathcal{U}(e, R)$. As such we do not employ a FSE on $\mathcal{U}(e, L)^\dagger$. This ambiguity also limits the application of a FSE to the CKM matrix. In this manner, the expansion we employ is most useful for expanding $\mathcal{U}(\nu, L)$ when a seesaw mechanism is the origin of the smallness of the neutrino masses.

We take $\mathcal{U}(e, L)^\dagger = \left(\vec{\sigma_3}^\dagger, \vec{\sigma_2}^\dagger, \vec{\sigma_1}^\dagger \right)$, with $\vec{\sigma}_i$ the orthonormal column eigenvectors diagonalizing $\mathcal{M}_e \mathcal{M}_e^\dagger$. The expanded $\mathcal{U}_{PMNS}^\star$ is of the form

$$\mathcal{U}_{PMNS}^\star = \begin{pmatrix} \vec{v}_3 \cdot \vec{\sigma}_3 & \vec{v}_2 \cdot \vec{\sigma}_3 & \vec{v}_1 \cdot \vec{\sigma}_3 \\ \vec{v}_3 \cdot \vec{\sigma}_2 & \vec{v}_2 \cdot \vec{\sigma}_2 & \vec{v}_1 \cdot \vec{\sigma}_2 \\ \vec{v}_3 \cdot \vec{\sigma}_1 & \vec{v}_2 \cdot \vec{\sigma}_1 & \vec{v}_1 \cdot \vec{\sigma}_1 \end{pmatrix},$$

$$= \begin{pmatrix} \vec{\rho}_3 \cdot \vec{\sigma}_3 & \vec{\rho}_2 \cdot \vec{\sigma}_3 & \vec{\rho}_1 \cdot \vec{\sigma}_3 \\ \vec{\rho}_3 \cdot \vec{\sigma}_2 & \vec{\rho}_2 \cdot \vec{\sigma}_2 & \vec{\rho}_1 \cdot \vec{\sigma}_2 \\ \vec{\rho}_3 \cdot \vec{\sigma}_1 & \vec{\rho}_2 \cdot \vec{\sigma}_1 & \vec{\rho}_1 \cdot \vec{\sigma}_1 \end{pmatrix} + \mathcal{O}(\delta\vec{\rho}_i, \delta_2\vec{\rho}_i). \tag{28}$$

This result makes clear that the first two right handed neutrinos that were integrated out in the FSE contribute to the leading order structure of the PMNS matrix. The leading order of the $\vec{\rho}_i$ eigenvectors only depended on $\vec{x}, \vec{y}$. As we have discussed, these neutrinos can be integrated out in sequence, and for Yukawa couplings that are degenerate the *lightest* two N_i can lead to the largest mass eigenvalues of the ν_L. In a mass degenerate case, they are more strongly coupled to the ℓ_{Lj}. The unitarity of the $\mathcal{U}_{PMNS}$ matrix is ensured when an expansion of this form is employed by normalizing the $\vec{v}_i$ eigenvectors order by order in the expansion. Conversely, if the FSE is used when the expansion parameters are not small, the constructed $\mathcal{U}_{PMNS}$ matrix is not necessarily unitary.

It is important to note that, in general, it is the relationship between the eigenvectors that determines $\mathcal{U}_{PMNS}$. Exact $\mathcal{TB}$ form at leading order in the FSE has a simple interpretation. It follows from the relationship between the eigenvector $\vec{\rho}_1$ and the σ_i being $\vec{\rho}_1 = (\vec{\sigma}_1 - \vec{\sigma}_2)/\sqrt{2}$ in our chosen phase convention.

III. (UN)RELATING FLAVOUR SYMMETRIES TO THE U_{PMNS} FORM

This formalism can be used to systematically explore symmetries that are consistent with TB form. Consider an exact $(\mathcal{TB})$ form of $\mathcal{U}_{PMNS}$ at leading order in the FSE. This is dictated by $\theta_{13} \equiv 0$ and θ_{23} maximal. We restrict the Yukawa vectors in this discussion to real values for simplicity. Including an unfixed s_{12} this gives (in a particular phase convention)

$$\mathcal{U}_{TB} = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -\frac{s_{12}}{\sqrt{2}} & \frac{c_{12}}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{s_{12}}{\sqrt{2}} & \frac{c_{12}}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}. \tag{29}$$

Assuming $\mathcal{U}_{PMNS} = \mathcal{U}_{TB}$

$$\begin{aligned}
\vec{\rho}_1 &= \frac{1}{\sqrt{2}}\left(\vec{\sigma}_1 - \vec{\sigma}_2\right), & \vec{\sigma}_1 &= \frac{1}{\sqrt{2}}(\vec{\rho}_1 + c_{12}\,\vec{\rho}_2 - s_{12}\,\vec{\rho}_3), \\
\vec{\rho}_2 &= \frac{c_{12}}{\sqrt{2}}(\vec{\sigma}_1 + \vec{\sigma}_2) + s_{12}\,\vec{\sigma}_3, & \vec{\sigma}_2 &= \frac{1}{\sqrt{2}}(-\vec{\rho}_1 + c_{12}\,\vec{\rho}_2 - s_{12}\,\vec{\rho}_3), \\
\vec{\rho}_3 &= -\frac{s_{12}}{\sqrt{2}}(\vec{\sigma}_1 + \vec{\sigma}_2) + c_{12}\,\vec{\sigma}_3, & \vec{\sigma}_3 &= s_{12}\,\vec{\rho}_2 + c_{12}\,\vec{\rho}_3.
\end{aligned} \tag{30}$$

Consider the "flavour" basis as an illustrative example of the FSE, where $\vec{\sigma}_1 = (0,0,1)$, $\vec{\sigma}_2 = (0,1,0)$, $\vec{\sigma}_3 = (1,0,0)$. In this case, $\mathcal{U}(e,L) = \mathrm{diag}(1,1,1)$ and $\mathcal{U}_{PMNS} = \mathcal{U}(\nu, L)$. The flavour basis is appealing in that it allows the solution for the perturbations to the $\vec{\rho}_i$ eigenvectors and the $\mathcal{U}_{PMNS}$ matrix to proceed simply through solving

$$\frac{v^2}{2\,\mu_a}\vec{x}\,\vec{x}^T + \frac{v^2}{2\,\mu_b}\vec{y}\,\vec{y}^T = \mathcal{U}_{TB}^{\star}\,\mathrm{diag}(0,\delta m_b, m_a + \delta m_a)\,\mathcal{U}_{TB}^{\dagger}, \tag{31}$$

$$\vec{x}\,\vec{x}^T + \frac{\mu_a}{\mu_b}\vec{y}\,\vec{y}^T = \frac{(m_a + \delta m_a)\,\mu_a}{v^2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} + \frac{\sqrt{2}\,\delta m_b\,\mu_a}{v^2}\begin{pmatrix} \sqrt{2}\,s_{12}^2 & s_{12}\,c_{12} & s_{12}\,c_{12} \\ s_{12}\,c_{12} & \frac{c_{12}^2}{\sqrt{2}} & \frac{c_{12}^2}{\sqrt{2}} \\ s_{12}\,c_{12} & \frac{c_{12}^2}{\sqrt{2}} & \frac{c_{12}^2}{\sqrt{2}} \end{pmatrix}.$$

Trivially one finds

$$\vec{x}^T = (0,-1,1)\,\sqrt{(m_a + \delta m_a)\,\mu_a}/v, \qquad \vec{y}^T = \left(\sqrt{2}\,s_{12}, c_{12}, c_{12}\right)\sqrt{\delta m_b\,\mu_b}/v. \tag{32}$$

It follows that in the flavour basis $\vec{x} \cdot \vec{y} = 0$ so $\delta m_a = 0$ and $(\delta\vec{\rho}_a, \delta\vec{\rho}_b, \delta\vec{\rho}_c) = (0,0,0)$. Now consider including a third neutrino eigenvalue, retaining the required perturbation. For a solution, $z_2 = z_3$ is required, and consequently $\delta_2 m_a^2 = 0$ as $\vec{x}\cdot\vec{z} = 0$. As a result the leading eigenvector $\vec{\rho}_a$ is unperturbed by integrating out both N_b, N_c in the flavour basis, one finds

$$\vec{z}^T = \pm\frac{\sqrt{2\mu_c\,\delta_2 m_c}}{v}\left(\sqrt{c_{12}^2 + \frac{\delta_2 m_b}{\delta_2 m_c}s_{12}^2}, \frac{1}{\sqrt{2}}\sqrt{\frac{\delta_2 m_b}{\delta_2 m_c}c_{12}^2 + s_{12}^2}, \frac{1}{\sqrt{2}}\sqrt{\frac{\delta_2 m_b}{\delta_2 m_c}c_{12}^2 + s_{12}^2}\right). \tag{33}$$

A $\mu \leftrightarrow \tau$ symmetry implemented on $\vec{y}$ and $\vec{z}$ is consistent with $\mathcal{TB}$ form of the PMNS matrix, as expected.

A.   Perturbative breaking of $\mathcal{TB}$ form

Now consider the breaking of $\mathcal{TB}$ form. The value of θ_{13} measured by the DAYA-Bay collaboration is in agreement with the global fit values given in Ref. [3], as such, we use the fit results to find the measured pattern of deviations in $\mathcal{TB}$ form. It is instructive to construct the following ratios of experimental values characterizing the deviations of $\mathcal{TB}$ form in each mixing angle. Using the small angle approximation

$$\frac{\tan^2(\delta\theta_{12})}{\sin^2(\delta\theta_{13})} = 0.02 \pm 0.32 \qquad \frac{\tan^2(\delta\theta_{23})}{\sin^2(\delta\theta_{13})} = 0.01 \pm 0.10 \qquad \frac{\tan^2(\delta\theta_{12})}{\tan^2(\delta\theta_{23})} = 2.0 \pm 36. \tag{34}$$

Here we have used the one sigma new reactor flux values of Ref. [3] and taken a $\pm$ symmetric one sigma error. Various breaking of $\mathcal{TB}$ form can be studied using the FSE and compared to these results. Consider the case that N_b retains a $\mu \leftrightarrow \tau$ flavour symmetry in its couplings to the charged leptons, but N_c breaks such a symmetry so that deviations in $\mathcal{TB}$ form are expected. Fix $\vec{z}^{T'} = \vec{z}^T + (0, \Delta_1, \Delta_2)$ with $\Delta_1 \neq \Delta_2$ and treat this breaking as a perturbation using the FSE. We use $\Delta m_{AB}^2 \simeq \Delta m_{AC}^2$ assuming a normal hierarchy. At leading order in the FSE the breaking of TB has the pattern

$$
\begin{aligned}
\frac{\tan^2(\delta\theta_{12})}{\sin^2(\delta\theta_{13})} &= 0, \\
\frac{\tan^2(\delta\theta_{23})}{\sin^2(\delta\theta_{13})} &= \frac{2\cos^2\theta_{\rho_b z} + \cos^2\theta_{\rho_c z}}{\cos^2\theta_{\rho_b z} + 2\cos^2\theta_{\rho_c z}}, \\
\frac{\tan^2(\delta\theta_{12})}{\tan^2(\delta\theta_{23})} &= 0.
\end{aligned}
\tag{35}
$$

As the range of the predicted value of $\tan^2(\delta\theta_{23})/\sin^2(\delta\theta_{13})$ is given by $[0.5, 2]$ at leading order in the FSE, this pattern of flavour breaking does not reproduce the data. In this way, particular mechanisms of the breaking of $\mathcal{TB}$ form can be falsified. One can perform the exercise of assuming $\mathcal{TB}$ form is broken in the flavour basis by N_b so that $\vec{y}^{T'} = \vec{y}^T + (0, \Delta_1, \Delta_2)$. Using the expansion one finds the pattern of deviations are distinct. These breakings of $\mathcal{TB}$ form are also correlated with mass perturbations in each case which are trivial to determine using this formalism.

We emphasize however that the relationships *between* the eigenvectors determine the form of the PMNS matrix in general. This is easy to demonstrate in more detail. Consider retaining a $\mu \leftrightarrow \tau$ symmetry imposed on $\vec{y}$ and $\vec{z}$ but deviating from the flavour basis, choosing $\vec{x}$, $\vec{y}$, $\vec{z}$ as above but unfixing $\vec{\sigma}_i$. At leading order in the expansion of $\mathcal{U}_{PMNS}$ one can solve for the condition that $\mathcal{TB}$ form is still obtained for general $\vec{\sigma}_i$. One finds that *only* the flavour basis for $\vec{\sigma}_i$ gives a valid solution at leading order in the expansion. This makes clear that $\mu \leftrightarrow \tau$ symmetry imposed on the Lagrangian alone is not related to $\mathcal{TB}$ form in a basis independent manner.

As a further example that $\mu \leftrightarrow \tau$ symmetry is also not unique or of particular physical significance in allowing $\mathcal{TB}$ form (with an appropriate choice on the σ_i eigenvectors), consider the following procedure. Choose a (e, μ, τ) symmetry on the first interaction eigenvector, ie $\vec{x}^T = (1, 1, 1)/\sqrt{3}$, and $\vec{y}^T = (1, 1, 0)/\sqrt{2}$ as a simple interaction eigenvector for N_b leading to an orthonormal eigenbasis at leading order, finding

$$
\mathcal{U}(\nu, L) = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}.
\tag{36}
$$

Solving directly for $\mathcal{U}(\nu, L)$ so that at leading order $\mathcal{TB}$ form is obtained, one finds

$$\mathcal{U}(e, L) = \begin{pmatrix} \frac{1}{6}\left(\sqrt{2} + 2\sqrt{3}\right) & \frac{1}{6}\left(\sqrt{2} - \sqrt{3} - \sqrt{6}\right) & \frac{1}{6}\left(\sqrt{2} - \sqrt{3} - \sqrt{6}\right) \\ \frac{1}{6}\left(\sqrt{2} - 2\sqrt{3}\right) & \frac{1}{6}\left(\sqrt{2} + \sqrt{3} - \sqrt{6}\right) & \frac{1}{6}\left(\sqrt{2} + \sqrt{3} + \sqrt{6}\right) \\ -\frac{\sqrt{2}}{3} & -\frac{2+\sqrt{3}}{3\sqrt{2}} & -\frac{\sqrt{2}}{3} + \frac{1}{\sqrt{6}} \end{pmatrix}. \tag{37}$$

This procedure can be used for any flavour symmetry chosen to fix $\vec{x}, \vec{y}$ in the FSE for the N_i.

We also note that the impact of sterile neutrinos weakly coupled to the SM on neutrino phenomenology can be systematically studied with this approach. For example, one can relate any measured value of a deviation from $\mathcal{TB}$ form to the particular Yukawa coupling vector of a single sterile neutrino, which can be shown to accommodate the value of θ_{13} reported by the Daya-Bay collaboration while the three right handed neutrinos partners of the SM fields give an exact $\mathcal{TB}$ form of the $\mathcal{U}_{PMNS}$ matrix.

IV. CONCLUSIONS

Flavour symmetries that are related to the structure of the $\mathcal{U}_{PMNS}$ matrix only in a particular basis choice of $\mathcal{U}(e, L)$ can lead to suspect physical conclusions. As an alternative to basis dependent symmetry studies, we have developed a perturbative expansion relating the measured masses of the neutrinos to the form of the PMNS matrix. This expansion offers a promising framework for broadly understanding the implications of the systematically improving experimental neutrino data, particularly in a normal hierarchy.

We have illustrated our approach in an example where the flavour basis was chosen, for the sake of familiarity, and then shown how the expansion can control the predictions of $\mathcal{TB}$ form being broken. However, the approach we outline can accommodate any basis choice. Indeed, it is the relationships between the eigenvectors that dictate the form of the $\mathcal{U}_{PMNS}$ matrix in a $\mathcal{U}(e, L)$ basis independent manner. This formalism can be employed in model building to attempt to determine a compelling origin of the eigenvector relationship that corresponds to $\mathcal{TB}$ form at leading order in the FSE. Further, as the breaking of $\mathcal{TB}$ form is now experimentally established due to the discovery of a non zero θ_{13} in Ref. [7], we expect the FSE to be of some phenomenological utility in falsifying mechanisms of the breaking of the $\mathcal{TB}$ form of the $\mathcal{U}_{PMNS}$ matrix, as the pattern of this breaking is further resolved experimentally in the years ahead.

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